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Melenberg, B.; Werker, B.J.M.

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Bertrand Melenberg and Bas J.M. Werker *
Tilburg University

Abstract

In this paper we reconsider the pricing of options in incomplete continuous time markets. We first discuss option pricing with idiosyncratic stochastic volatility. This leads, of course, to an averaged Black-Scholes price formula. Our proof of this result uses a new formalization of idiosyncraticy which encapsulates other definitions in the literature. Our method of proof is subsequently generalized to other forms of incompleteness and systematic (i.e. non-idiosyncratic) information. Generally this leads to an option pricing formula which can be expressed as the average of a complete markets formula.

Keywords: Idiosyncraticy, Incomplete markets, Option pricing, Stochastic volatility

1 Introduction

Continuous time option pricing has evolved along several lines. One way is to formulate a general equilibrium asset pricing model which endogenously determines the stochastic processes followed by the equilibrium price of any financial asset, including options. See, for example, Cox, Ingersoll and Ross (1985) and Hull and White (1987). An other way is the arbitrage pricing approach as applied by, for example, Black and Scholes (1973) and Cox and Ross (1976), and formalized by Harrison and Kreps (1979), followed by Harrison and Pliska (1981) and Huang (1985). In this approach one starts with the price process of the underlying assets. The existence of a so-called "equivalent martingale measure", which turns the price process of the underlying assets (after adjustment for discounting) into a martingale, ensures that arbitrage opportunities are excluded. Each equivalent martingale measure can be used to calculate a candidate price for options (and other derivative securities).

*CentER and Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. We would like to thank Peter Bossaerts for several useful remarks.

But options (and other derivative securities) are only "priced by arbitrage" when all possible equivalent martingale measures yield the same candidate price.

When markets are complete, there will be only one equivalent martingale measure, so that options (and other derivative securities) can be priced by arbitrage. However, when markets are incomplete, pricing by arbitrage is generally impossible, due to the existence of different equivalent martingale measures which may yield different candidate prices for options (and other derivative securities). Only by imposing additional restrictions pricing by arbitrage might become possible.

To see how additional restrictions might be imposed in case of an incomplete markets model, consider Hull and White (1987). These authors study a Black-Scholes type model for the underlying asset, but with a volatility that depends on some underlying stochastic process, so that their model may be referred to as a Black-Scholes model with stochastic volatility. Hull and White (1987) calculates option prices from an equilibrium asset pricing model, under the assumption that the beta of the stochastic process driving the volatility (of the underlying asset) with respect to aggregate consumption is equal to zero. In terms of the arbitrage pricing approach the Hull and White option pricing idea can be reformulated as follows. Using He and Pearson (1991) the Radon-Nikodym derivative with respect to the true underlying probability measure of every equivalent martingale measure can be decomposed into two components. The first component is fully determined by the underlying asset price process; it is the same for all equivalent martingale measures. The second component is (more or less) determined by the process driving the volatility. The equivalent martingale measures only differ through the second term. In terms of this decomposition the "beta is zero" condition, as imposed by Hull and White, boils down to a restriction on the second term of the decomposition, namely that it equals one. This restriction makes pricing by arbitrage possible. However, fixing the second term in the decomposition in some other way, also makes pricing by arbitrage possible. The fixing of the second term is essential, not that it is fixed to one.

The contribution of this paper is to exploit the idea of decomposing the equivalent martingale measures in such a way that fixing some component of the decomposition reduces the set of equivalent martingale measures sufficiently well so that pricing by arbitrage becomes possible. However, instead of first decomposing the equivalent martingale measures [similar to He and Pearson (1991)] and to fix some component in the decomposition, we will fix what we will call the price of an information structure. Here an information structure is some subfiltration of the filtration by which the information is revealed in the economy. As an example think of the information generated by the volatility. The definition of the price of an information structure is such that, when all equivalent martingale measures that have the same information structure price yield the same "conditional option price" (conditional on the information generated by the subfiltration), then the option can be priced by arbitrage. The resulting option price then depends on the "conditional option price" (which is unique by construction) and the price of the information structure (which is not restricted by the given price processes).

Heuristically speaking, one might say that we reduce the option pricing problem to a complete market option pricing problem by introducing the extra information. To derive the actual option price therefrom we introduce the price of the completing information structure. An advantage of this method is that all methods to solve the complete market option pricing problem can be used in the first step (e.g., duplicating portfolio's, Feynmann-Kac partial differential equations, and Girsanov transformations). We present our techniques in a rather abstract way to allow for general (i.e. not necessarily Markovian Itô processes) price processes. However, we give numerous examples to show how the techniques can be applied. These examples include, among others, the model of Hull and White (1987), with and without correlation between asset prices and stochastic volatility, and the model of Cox, Ingersoll, and Ross (1985) with state dependent volatility.

Our definition of the price of an information structure is such that a zero price of an information structure (i.e. no risk premium), which is generated by some underlying stochastic process, corresponds to the restriction “beta is zero”, where beta is now defined (since markets are incomplete and aggregate consumption is not modeled) with respect to the equivalent martingale measure. In this situation we will call the information structure idiosyncratic, which we will show to be in line with the use of this term in the literature.

The remainder of the paper is organized as follows. Since the concept of idiosyncratic information plays an important role in the finance literature, we first introduce in section 2 our definition of idiosyncratic information and relate it to other definitions employed in the literature. In this section the concept "price of an information structure" is not used. In section 3 we derive, under rather general conditions, the price of an option on an asset with idiosyncratic volatility. In section 4 we introduce the concept "price of an information structure" and show how it can be used in option pricing. Several examples will illustrate the theory. Section 5 concludes and discusses some possible extensions.

2 Idiosyncratic information

We start this section by introducing the general setup for the economy. This also fixes some notation. Consider an economy in which uncertainty is modeled by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The time horizon in this economy is $[0, T]$ and information is revealed according to the filtration $\{\mathcal{F}_t : \ t \in [0, T]\}$, which is assumed to satisfy the usual conditions, e.g. the filtration is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$. In this economy several assets are traded. We have a stock S, a bond B with maturity T, and a money market account M. It is understood that other assets are traded as well, but we will focus on S, B, and M. The price of the asset S at time t will be denoted by S_t, and similarly for B and M. We assume that the process given by (S_t, B_t, M_t) is adapted to the filtration $(\mathcal{F}_t)$, i.e. we assume that asset prices are observed. Asset M satisfies $M_0 = 1$ and

$$\mathrm{d}M_t = M_t r_t \mathrm{d}t, \tag{2.1}$$

where $\{r_t : \ t \geq 0\}$ denotes the (possibly stochastic) instantaneous interest rate prevailing at time t. Consequently, $M_t = \exp\left\{\int_0^t r_s \mathrm{d}s\right\}$. With respect to the asset prices we assume the existence of at least one equivalent martingale measure $\mathbb{Q}$. I.e. $\mathbb{Q}$ is some probability measure equivalent to $\mathbb{P}$ such that on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : \ t \in [0,T]\}, \mathbb{Q})$ the discounted price processes S_t/M_t and B_t/M_t are $\mathbb{Q}$-martingales. This assumption is a natural way to exclude arbitrage possibilities in a frictionless market [see, e.g., Harrison and Kreps (1979)]. There may be more than one $\mathbb{Q}$ which will happen when the market is incomplete. This concludes our basic setup. Throughout we denote expectations under $\mathbb{Q}$ by $E_{\mathbb{Q}}$, and expectations under $\mathbb{P}$ by E.

As an illustration and motivation consider the following specialization of the model, i.e., (2.1) [with $r_t = r(Z_t, S_t)$] combined with

$$\begin{aligned} \mathrm{d}S_t &= \mu(Z_t, S_t)\mathrm{d}t + \sigma(Z_t, S_t)\mathrm{d}W_t^{(1)}, & (2.2)\\ \mathrm{d}Z_t &= \nu(Z_t)\mathrm{d}t + \tau(Z_t)\mathrm{d}W_t^{(2)}. & (2.3) \end{aligned}$$

Here $(W^{(1)}, W^{(2)})$ forms a bivariate standard Brownian Motion (with $W_t^{(1)}$ and $W_t^{(2)}$ uncorrelated) and Z_t is some state variable that is allowed to influence the instantaneous drift and volatility of S_t. Now we have $\mathcal{F}_t = \sigma(S_s, Z_s : \ 0 \leq s \leq t) = \sigma(W_s^{(1)}, W_s^{(2)} : \ 0 \leq s \leq t)$. Model (2.1)–(2.3) encompasses the model employed by Hull and White (1987) which can be obtained by choosing

$$\begin{aligned} r_t &= r,\\ \mu(Z_t, S_t) &= \mu(Z_t) S_t,\\ \sigma(Z_t, S_t) &= \sqrt{Z_t} S_t,\\ \nu(Z_t) &= \nu Z_t,\\ \tau(Z_t) &= \tau Z_t. \end{aligned}$$

Now, as a consequence of Proposition 1 of He and Pearson (1991), the set of equivalent martingale measures Q is fully characterized by the following Radon-Nikodym derivatives

$$\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = \mathcal{E}(X)_T \mathcal{E}(Y_\lambda)_T, \tag{2.4}$$

where $\mathcal{E}(X)_t$ is the exponential of

$$\mathrm{d}X_t = \kappa_t \mathrm{d}W_t^{(1)},$$

with $\kappa_t = -(\mu(Z_t, S_t) - r_t S_t)/\sigma(Z_t S_t)$ and $\mathcal{E}(Y_\lambda)_t$ denotes the exponential of

$$\mathrm{d}Y_{\lambda t} = \lambda_t \mathrm{d}W_t^{(2)},$$

where λ_t defines a progressively measurable process satisfying $\int_0^T \lambda_t^2 \mathrm{d}t < \infty$ $(a.s)$. Notice that

$$\begin{aligned} \mathcal{E}(X)_T &= \exp(\int_0^T \kappa_t \mathrm{d}W_t^{(1)} - \frac{1}{2}\int_0^T \kappa_t^2 \mathrm{d}t), \\ \mathcal{E}(Y_\lambda)_T &= \exp(\int_0^T \lambda_t \mathrm{d}W_t^{(2)} - \frac{1}{2}\int_0^T \lambda_t^2 \mathrm{d}t). \end{aligned}$$

Clearly, without further restrictions, pricing by arbitrage following the lines of Harrison and Kreps (1979) will in general be impossible in the model (2.1)–(2.3), since the class of equivalent martingale measures is not a singleton. Similar to Hull and White (1987) we could impose as additional restriction that the beta of the state process Z is equal to zero. In the equilibrium model of Hull and White (1987) the beta is defined with respect to aggregate consumption. Without consumption we can consider the beta of Z with respect to $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ (for some given λ)[1]. Then β satisfies

$$\beta_t = \frac{\tau(Z_t)\lambda_t}{[\kappa_t^2 + \lambda_t^2]\mathcal{E}(X)_t \mathcal{E}(Y_\lambda)_t} \tag{2.5}$$

The condition $\beta_t = 0$ (for all t) corresponds to $\lambda_t = 0$ (for all t), assuming $\tau(Z_t) > 0$ and using $\mathcal{E}(Y_\lambda)_t > 0$. The restriction $\lambda_t = 0$ (for all t) means in terms of $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$:

$$\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = \mathcal{E}(X)_T \mathcal{E}(Y_0)_T = \mathcal{E}(X)_T \cdot 1. \tag{2.6}$$

Thus the restriction $\beta_t = 0$ or $\lambda_t = 0$ for all t restricts the set of equivalent martingale measures to a singleton, and makes pricing by arbitrage possible.

The restriction $\lambda_t = 0$ or condition (2.6) can be interpreted as a restriction on the information generated by Z_t (or $W_t^{(2)}$). Let $\tilde{\mathcal{F}}_t$ denote $\sigma(Z_s;\ 0 \le s \le t) = \sigma(W_s^{(2)};\ 0 \le s \le t)$. Then (2.6) can be reformulated as

$$E\{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} | \mathcal{F}_t\} = E\{\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} | \mathcal{F}_t \vee \tilde{\mathcal{F}}_T\}, \tag{2.7}$$

for all $t \in [0, T]$. Notice that as long as $\lambda_t = 0$ for all t (a.s.) the left-hand and right-hand side of (2.7) are equal to each other, whereas if this restriction on λ_t is not satisfied, the right-hand side will be different, violating (2.6). The condition that the beta of Z is equal to zero is usually referred to as "Z_t has zero systematic risk". In view of (2.7) we will say that "the information generated by Z_t is idiosyncratic".

We now want to formalize the concept of zero systematic risk or idiosyncratic information in terms of the general model as introduced at the beginning of this section. For that purpose we shall use the alternative formulation of the "beta=0"–condition as given in equation (2.7). Thus our definition of idiosyncratic information becomes

[1]See Back (1991) for the relation between these two definitions.

Definition 2.1 *An information structure* $\{\tilde{\mathcal{F}}_t:\ t \geq 0\}$ *is called idiosyncratic with respect to* $\mathbb{Q}$ *and up to time* T *if*

1. $\{\tilde{\mathcal{F}}_t:\ t \geq 0\}$ *constitutes a filtration such that, for all* $t \geq 0$, $\tilde{\mathcal{F}}_t \subset \mathcal{F}_t$ *and*

2. *for all* $t \in [0, T]$
$$E\left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right\} = E\left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \vee \tilde{\mathcal{F}}_T \right\}. \tag{2.8}$$

In the general case different $\mathbb{Q}$-s may satisfy this definition, so that pricing by arbitrage is still not possible in general, when using Definition 2.1 to restrict the class of equivalent martingale measures. However, as we will see in the next section, there are models [encompassing model (2.1)–(2.3)] where imposing the additional restriction of Definition 2.1 makes pricing by arbitrage possible.

Definition 2.1 can be seen as a definition of idiosyncratic information on a macro level. In the literature one can also find descriptions of idiosyncratic information on a micro level. For instance, Merton (1990) considers an individual consumer maximizing a von Neumann-Morgenstern expected intertemporally additive utility function subject to a budget constraint, employing relations (2.1)–(2.3) to describe the dynamics. Merton (1990) assumes constant coefficients in (2.1)–(2.3), thus

$$r_t = r, \quad \mu_t(Z_t, S_t) = \mu S_t, \quad \sigma(Z_t, S_t) = \sigma S_t. \tag{2.9}$$

Merton (1990) allows for instantaneous correlation between $W_t^{(1)}$ and $W_t^{(2)}$. The state variable Z_t is then called idiosyncratic if the instantaneous correlation between Z_t and S_t (or between $W_t^{(1)}$ and $W_t^{(2)}$) is equal to zero. This definition is clearly not equivalent with the restriction $\lambda_t = 0$ (for all t) as in (2.6). However, combining He and Pearson (1991) and (2.9), it follows that the individual consumer's optimization problem can be solved by assuming $\mathcal{E}(Y_\lambda) = \mathcal{E}(Y_0) = 1$. In the terminology of He and Pearson, the choice $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = \mathcal{E}(X)$ is the "minimax local martingale measure", when Z_t is idiosyncratic according to the definition of Merton. If, in addition, Z_t is idiosyncratic for all consumers in the economy, then, since $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = \mathcal{E}(X)$ is the "minimax local martingale measure" for all consumers, in equilibrium we also will have $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = \mathcal{E}(X)$, i.e., $\mathcal{E}(Y_\lambda) = \mathcal{E}(Y_0) = 1$. In other words, idiosyncraticy for all consumers on a micro level according to the definition of Merton (1990) (and given his model) implies idiosyncracy on a macro level according to Definition 2.1.

We conclude this section by some consequences of idiosyncraticy of information. As a first result we have

Lemma 2.2 *Suppose that* X *is measurable with respect to* $\mathcal{F}_t \vee \tilde{\mathcal{F}}_T$ *where* $\{\tilde{\mathcal{F}}_t\}$ *is idiosyncratic up to time* T. *Then*

$$E_{\mathbb{Q}}\left\{ X \middle| \mathcal{F}_t \right\} = E\left\{ X \middle| \mathcal{F}_t \right\}.$$

PROOF: Using Bayes rule [see, e.g., Karatzas and Shreve (1988)] we obtain

$$\begin{aligned} E_{\mathbb{Q}}\{X|\mathcal{F}_t\} &= E\left\{ X\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}/E\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|\mathcal{F}_t\}\middle|\mathcal{F}_t\right\} \\ &= E\left\{ XE\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|\mathcal{F}_t\vee\tilde{\mathcal{F}}_T\}/E\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|\mathcal{F}_t\}\middle|\mathcal{F}_t\right\} \\ &= E\{X|\mathcal{F}_t\}.\Box \end{aligned}$$

This lemma will be needed in Section 3 and will be extended to the case of non-idiosyncratic information in Section 4. A second result is an immediate consequence of this lemma. It states that no risk premium will be paid on financial claims which have idiosyncratic risk only.

Corollary 2.3 *If X denotes the pay-off of a financial claim and X/S_{0T} would be $\tilde{\mathcal{F}}_T$-measurable, then the price of X is simply its expected discounted payoff (under $\mathbb{P}$!).*

3 Option pricing with idiosyncratic volatility

In Section 2 we presented the concept of idiosyncratic information structures. In this section we will show how this concept can be used to solve the option pricing problem in case the underlying asset exhibits stochastic volatility. As a consequence we obtain the Merton stochastic interest rate and the Hull–White stochastic volatility option pricing formulae under rather primitive assumptions on the stochastic process of the underlying asset. In Section 4 we will generalize this approach to other incomplete markets where the factor inducing the incompleteness is not necessarily idiosyncratic.

Let S be the price process of an asset on which a European option with payoff $\max\{0, S_T - K\}$ at time T is available[2]. Let B denote the price of a default-free pure discount bond with the same maturity as the option, i.e. $B_T = 1$. Finally denote the money market account again by M.

Assumption A Suppose that the following conditions are satisfied.

1. S and B have continuous sample paths[3],

2. the risk associated with the volatility in $\log S$ and $\log B$ is idiosyncratic. More precisely

$$\begin{aligned} \tilde{\mathcal{F}}_t &= \sigma\left(\langle\log S, \log S\rangle_s : \ s\in[0,t]\right)\vee \\ &\quad \sigma\left(\langle\log S, \log B\rangle_s : \ s\in[0,t]\right)\vee \\ &\quad \sigma\left(\langle\log B, \log B\rangle_s : \ s\in[0,t]\right) \end{aligned} \tag{3.1}$$

[2]We will focus on the pricing of European call options, but it should be obvious from the rest of the paper that the techniques developed easily allow for the pricing of more complicated derivatives.

[3]See Huang (1985) for sufficient conditions on the underlying information structure to ensure continuity of asset prices' sample paths.

is idiosyncratic with respect to $\mathbb{Q}$ up to time T, and

3. the continuous local martingale parts of $\log S$ and $\log B$ are $(\mathbb{P}, \mathcal{F}_t \vee \tilde{\mathcal{F}}_T)$-local martingales as well.

Let us discuss each of these assumptions shortly. The first assumption rules out the possibility of jumps in the underlying asset's price process. Therefore the results in this section cannot be used to derive the option prices in case the underlying asset has discontinuous sample paths as in Merton (1976). We will come back to this point in Section 5. The second assumption states that the stochastic volatility in the underlying asset S and the bond B is idiosyncratic. As follows from the next theorem, this assumption reduces the set of possible equivalent martingale measures in such a way that pricing by arbitrage becomes possible, i.e. all remaining equivalent martingale measures will yield the same option price. The final assumption is a version of the assumption in Hull and White (1987) which roughly states that the stochastic volatility process may not be influenced by the asset price (see Example 3.2 below). Assumption A.3 is also comparable to Assumption 2 of Amin and Ng (1993). In Section 4 we will show how to handle stochastic volatility models where this assumption is not satisfied.

We can now state the main theorem of this section.

Theorem 3.1 *Under Assumption A the time t price of a European option with maturity T and exercise price K is given by*

$$E\left\{ BS(S_t; V_T - V_t, KB_t) \middle| \mathcal{F}_t \right\}, \tag{3.2}$$

where

$$\begin{aligned} V_t &= \langle \log S/B, \log S/B \rangle_t \\ BS(S; V, \tilde{K}) &= S\Phi(d) - \tilde{K}\Phi(d - \sqrt{V}), \\ d &= \frac{\log(S/\tilde{K}) + \frac{1}{2}V}{\sqrt{V}}. \end{aligned} \tag{3.3}$$

Proof: By assumption $(S/M, B/M)$ constitutes a continuous $(\mathbb{Q}, \mathcal{F}_t)$-martingale. Define the quadratic variation processes

$$\begin{aligned} Q_t^{SS} &= \langle \log S/M, \log S/M \rangle_t, \\ Q_t^{BS} &= \langle \log B/M, \log S/M \rangle_t, \text{ and} \\ Q_t^{BB} &= \langle \log B/M, \log B/M \rangle_t, \end{aligned}$$

then, using Itô's Lemma, we obtain that

$$\begin{aligned} X_t^S &= \log S_t/M_t - \log S_0 + \frac{1}{2} Q_t^{SS} \text{ and} \\ X_t^B &= \log B_t/M_t - \log B_0 + \frac{1}{2} Q_t^{BB} \end{aligned}$$

are continuous $(\mathbb{Q},\mathcal{F}_t)$-local martingales with $\langle X^S,X^S\rangle_t = Q_t^{SS}$ and $\langle X^B,X^B\rangle_t = Q_t^{BB}$. We will show that both X^S and X^B are $\mathbb{Q}$-local martingales with respect to the filtration $\mathcal{F}_t \vee \tilde{\mathcal{F}}_T$ as well. To be precise, we will show that X^S and X^B are continuous local martingales under $\mathbb{Q}$ with respect to the filtration $\mathcal{G}$ given by $\mathcal{G}_t = \mathcal{F}_t \vee \tilde{\mathcal{F}}_T$. The argument for both X^S and X^B is the same, so we will only consider X^S. From Girsanov's Theorem we know that there exists a finite variation process A^S such that $X_t^S + A_t^S$ defines a continuous $(\mathbb{P},\mathcal{F}_t)$-local martingale and hence equals the continuous local martingale part of $\log S/M$ and of $\log S$. By Assumption A.3 it is a continuous $(\mathbb{P},\mathcal{F}_t \vee \tilde{\mathcal{F}}_T)$-local martingale as well. Using Girsanov's Theorem once more, but now with the filtration $\mathcal{F}_t \vee \tilde{\mathcal{F}}_T$ which does not alter the likelihood ratio by the idiosyncraticy assumption on $\tilde{\mathcal{F}}_T$, we obtain that X^S is a continuous $(\mathbb{Q},\mathcal{F}_t \vee \tilde{\mathcal{F}}_T)$-local martingale.

From the local martingales X^S and X^B we construct orthogonal local martingales by

$$\begin{aligned} Y_t^S &= X_t^S \text{ and} \\ Y_t^B &= X_t^B - \int_0^t \frac{\mathrm{d}\langle X^B,X^S\rangle_s}{\mathrm{d}\langle X^S,X^S\rangle_s}\mathrm{d}X_s^S. \end{aligned}$$

From Knight's Theorem [a multivariate version of the Dambis, Dubins-Schwarz Theorem, see, e.g., Revuz and Yor (1991)] we obtain that[4]

$$W_t^S = Y_{T_t^S}^S \text{ and } W_t^B = Y_{T_t^B}^B,$$

with $T_t^S = \inf\{s : \langle Y^S,Y^S\rangle_s > t\}$ and $T_t^B = \inf\{s : \langle Y^B,Y^B\rangle_s > t\}$, define independent $\mathbb{Q}$-Brownian Motions. By definition of $\tilde{\mathcal{F}}_T$ we know that Q^{SS}, Q^{BS}, and Q^{BB} are $\tilde{\mathcal{F}}_T$ measurable and so are T^S and T^B. Consequently,

$$\begin{aligned} X_T^S &= W_{Q_T^{SS}}^S \text{ and} \\ X_T^B &= W_{\langle Y^B,Y^B\rangle_T}^B + \int_0^T \frac{\mathrm{d}Q_s^{BS}}{\mathrm{d}Q_s^{SS}}\mathrm{d}W_{Q_s^{SS}}^S \end{aligned}$$

consist in fact of deterministic (i.e. $\mathcal{F}_0 \vee \tilde{\mathcal{F}}_T$-measurable) integrals of $\mathbb{Q}$-Brownian Motions. Trivial calculations now yield that the distribution of $(\log S_T/M_T, \log M_T)$, conditional on $\mathcal{F}_t \vee \tilde{\mathcal{F}}_T$ and under $\mathbb{Q}$, is

$$N\left(\begin{bmatrix} \log S_t/M_t - \frac{1}{2}(Q_T^{SS} - Q_t^{SS}) \\ -\log B_t/M_t + \frac{1}{2}(Q_T^{BB} - Q_t^{BB}) \end{bmatrix} ; \begin{bmatrix} Q_T^{SS} - Q_t^{SS} & -[Q_T^{BS} - Q_t^{BS}] \\ -[Q_T^{BS} - Q_t^{BS}] & Q_T^{BB} - Q_t^{BB} \end{bmatrix}\right). \tag{3.4}$$

We are now in a position to calculate

$$M_t E_{\mathbb{Q}}\left\{\left.\frac{\max\{0,S_T-K\}}{M_T}\right|\mathcal{F}_t\right\} = M_t E_{\mathbb{Q}}\left\{\left. E_{\mathbb{Q}}\left\{\max\{0,S_T/M_T - K/M_T\}\,|\,\mathcal{F}_t \vee \tilde{\mathcal{F}}_T\right\}\right|\mathcal{F}_t\right\}.$$

[4] Formally we need the condition $\langle Y^S,Y^S\rangle_\infty = \langle Y^B,Y^B\rangle_\infty = \infty$ but this can be met by constructing a large enough probability space for our model.

and Lemma A.1 (see appendix) together with Lemma 2.2 gives the desired result. □

In its general form the previous theorem allows us to calculate option prices under the assumption that the stochastic volatility is idiosyncratic. We will now show how this theorem specializes to several well-known results from the literature. Note that any equilibrium model that satisfies our assumptions will yield an option price given by Theorem 3.1. We will assume throughout that the underlying price process S does not exhibit jumps. In all cases it is straightforward to verify that Assumption A.2 and A.3 are met.

We start by considering the standard Black-Scholes world.

Example 3.1 Consider the economy described at the beginning of this section. Assume that the instantaneous interest rate r is constant and $\langle \log S, \log S \rangle_t = \sigma^2 t$. Then the price of an European option with expiration date T and exercise price K is

$$BS(S_t; \sigma^2(T-t), K\exp(-r(T-t))),$$

where BS is given by (3.3). □

Note that we did not make any explicit assumption on market completeness, i.e. there might be more Brownian motions than the one governing the underlying asset's price. We showed that under all possible equivalent martingale measures, satisfying Assumption A, the option price is the same. Hence for this restricted set of equivalent martingale measures, pricing by arbitrage of the option is possible. See also Example 3.3

Let us now consider the model described in Hull and White (1987).

Example 3.2 Consider the economy described at the beginning of this section. Assume that the instantaneous interest rate r is constant and that the asset price proces is given by

$$\begin{aligned} \mathrm{d}S_t &= \varphi(S_t, \sigma_t) S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t^{(1)}, \\ \mathrm{d}\sigma_t^2 &= \mu(\sigma_t)\sigma_t^2 \mathrm{d}t + \xi(\sigma_t)\sigma_t^2 \mathrm{d}W_t^{(2)}, \end{aligned}$$

where $W^{(1)}$ and $W^{(2)}$ are independent Brownian motions. If the stochastic volatility σ_t^2 is idiosyncratic, then the price of a European call option with expiration date T and exercise price K is given by

$$E\{BS(S_t; \int_t^T \sigma_s^2 \mathrm{d}s, K\exp(-r(T-t)))|\mathcal{F}_t\}.$$

Notice that, analogous to (2.4), the set of equivalent martingale measures $\mathbb{Q}$ is fully characterized by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}(X)_T \mathcal{E}(Y_\lambda)_T,$$

with

$$\begin{aligned} \mathrm{d}X_t &= \kappa_t \mathrm{d}W_t^{(1)}, \quad \kappa_t = -\frac{\varphi(S_t, \sigma_t) - r_t}{\sigma_t} \\ \mathrm{d}Y_{\lambda t} &= \lambda_t \mathrm{d}W_t^{(2)}, \end{aligned}$$

where λ_t is progressively measurable with $\int_0^T \lambda_t^2 \mathrm{d}t < \infty$ $(a.s)$. The additional restriction that the volatility σ_t^2 is idiosyncratic is equivalent to the restriction that $\lambda_t = 0$, $t \in [0, T]$ $(a.s)$. □

As in Hull and White (1987) we find that the option price is the expectation over the Black–Scholes formula. In Section 4 we will show that option prices in incomplete markets are generally an expectation over a "complete markets option pricing formula".

In Example 3.2 conditioning out the stochastic volatility implies that one essentially has only one Brownian motion left and hence a complete market. The following example shows that Theorem 3.1 can also be applied if more Brownian motions remain after conditioning on the stochastic volatility process.

Example 3.3 Consider the economy described at the beginning of this section. Assume that the instantaneous interest rate r is constant and that the asset price proces is given by

$$\begin{aligned} \mathrm{d}S_t &= \varphi(S_t, Z_t) S_t \mathrm{d}t + \sigma_1(Z_t) S_t \mathrm{d}W_t^{(1)} + \sigma_2(Z_t) S_t \mathrm{d}W_t^{(2)}, \\ \mathrm{d}Z_t &= \mu(Z_t)\mathrm{d}t + \xi(Z_t)\mathrm{d}W_t^{(3)}, \end{aligned}$$

where $(W^{(1)}, W^{(2)}, W^{(3)})$ is a trivariate Brownian motion with $W^{(3)}$ independent of $(W^{(1)}, W^{(2)})$ and correlation ρ between $W^{(1)}$ and $W^{(2)}$. If the information generated by Z_t is idiosyncratic, then the price of a European call option with expiration date T and exercise price K is given by

$$E\{BS(S_t; \int_t^T \sigma_1^2(Z_s) + \sigma_2^2(Z_s) + 2\rho\sigma_1(Z_s)\sigma_2(Z_s)\mathrm{d}s, K\exp(-r(T-t)))|\mathcal{F}_t\}.$$

Notice that in this case the set of equivalent martingale measures $\mathbb{Q}$ is fully described by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}(X)_T \mathcal{E}(Y_\lambda)_T,$$

with

$$\begin{aligned} \mathrm{d}X_t &= \kappa_t^{(1)} \mathrm{d}W_t^{(1)} + \kappa_t^{(2)} \mathrm{d}W_t^{(2)} \\ \begin{pmatrix} \kappa_t^{(1)} \\ \kappa_t^{(2)} \end{pmatrix} &= -\frac{\varphi(S_t, \sigma_t) - r_t}{\sigma_1^2(Z_t) + \sigma_2^2(Z_t)} \begin{pmatrix} \sigma_1(Z_t) \\ \sigma_2(Z_t) \end{pmatrix}, \\ \mathrm{d}Y_{\lambda t} &= \sum_{j=1}^{3} \lambda_t^{(j)} \mathrm{d}W_t^{(j)}, \end{aligned}$$

where $(\lambda_t^{(1)}, \lambda_t^{(2)}, \lambda_t^{(3)})$ is progressively measurable with $\int_0^T \sum_{j=1}^3 (\lambda_t^{(j)})^2 \mathrm{d}t < \infty$ $(a.s)$ and where, $t \in [0, T]$,

$$\kappa_t^{(1)} \lambda_t^{(1)} + \kappa_t^{(2)} \lambda_t^{(2)} = 0 \ (a.s).$$

The assumption that the information generated by Z_t is idiosyncratic forces $\lambda_t^{(3)} = 0,\ t \in [0, T]$ $(a.s)$. But $\lambda^{(1)}$ and $\lambda^{(2)}$ are not further restricted by the idiosyncratici assumption. Despite of this the option can be priced by arbitrage, since, loosely speaking, conditional on Z_t,

$$\sigma_1(Z_t) S_t \mathrm{d}W_t^{(1)} + \sigma_2(Z_t) S_t \mathrm{d}W_t^{(2)}$$

can be represented by

$$\sqrt{\sigma_1^2(Z_t) + \sigma_2^2(Z_t)}\, S_t \mathrm{d}W_t,$$

with W_t a single Brownian motion. From such a representation a unique option price follows. □

As a further illustration of our general formula consider the stochastic interest rate option pricing formula of Merton (1990).

Example 3.4 Consider the economy described at the beginning of this section. Assume that

$$\mathrm{d} \log S_t / B_t = \tilde{\mu}_t \mathrm{d}t + \tilde{\sigma} \mathrm{d}W_t,$$

where W is some Brownian motion and assume that $\langle \log S, \log S \rangle_t$, $\langle \log S, \log B \rangle_t$, and $\langle \log B, \log B \rangle_t$ are deterministic. Then the price of a European call option with expiration date T and exercise price K is given by

$$BS(S_t; \tilde{\sigma}^2 (T - t), K B_t),$$

where B_t is the price of a pure discount bond maturing at time T. □

As is well-known we also find that it is the variance of the discounted price process $\log S_t / B_t$ that is relevant for the option prices. Of course, if $\tilde{\sigma}_t^2$ is a deterministic function of time, $\tilde{\sigma}^2 (T - t)$ should be replaced with $\int_t^T \tilde{\sigma}_s^2 \mathrm{d}s$.

4 Option pricing using systematic information

In the former section we derived an option pricing formula under the assumption that the volatility bears idiosyncratic risk only and does not influence the semimartingale decomposition of the asset price process (under the true probability measure $\mathbb{P}$). In this section we will show how to generalize this result. The generalization will be in two directions. First of all we do not necessarily consider the information structure generated by the stochastic volatility as being idiosyncratic. Secondly, we will allow the information structure to have a price unequal to zero (idiosyncratic information will have price zero).

As a motivation for the definitions and lemma's in this section, let us recall the way of reasoning in the proof of Theorem 3.1. The key idea was to use the law of iterated expectations to write

$$M_t E_{\mathbb{Q}} \{\max\{0, S_T - K\}/M_T | \mathcal{F}_t\} =$$

$$E_{\mathbb{Q}} \left\{ M_t E_{\mathbb{Q}} \left\{ \max\{0, S_T - K\}/M_T | \mathcal{F}_t \vee \tilde{\mathcal{F}}_T \right\} \middle| \mathcal{F}_t \right\}. \tag{4.1}$$

Using a convenient choice for $\tilde{\mathcal{F}}_t$ (in Section 3 this was the information in the stochastic volatility) the inner expectation in (4.1) was evaluated by showing that, under every $\mathbb{Q}$ satisfying the conditions of Assumption A.2 and A.3, the conditional distribution of S_T given $\mathcal{F}_t \vee \tilde{\mathcal{F}}_T$ does not depend on $\mathbb{Q}$. This gives a unique value to the inner expectation in (4.1) and hence, using Lemma 2.2, an option price by arbitrage follows. This approach can be pursued much more generally and this is the objective of this section. We will show that option prices are generally equal to the expectation over a complete markets option pricing formula like in (3.2). If the information structure involved is not idiosyncratic, we must use a weighted expectation.

We will begin with defining what we will understand under the price of an information structure. Recall the $\mathbb{Q}$ denotes the equivalent martingale measure.

Definition 4.1 *An information structure $\{\tilde{\mathcal{F}}_t : \ t \geq 0\}$ has (time T) price $\tilde{R}_t^T$ under $\mathbb{Q}$ if*

1. *$\{\tilde{\mathcal{F}}_t : \ t \geq 0\}$ constitutes a filtration such that, for all $t \geq 0$, $\tilde{\mathcal{F}}_t \subset \mathcal{F}_t$ and*
2. *$\tilde{R}_t^T$ defined by*
$$\frac{E\left\{ \frac{d\mathbb{Q}}{d\mathrm{P}} \middle| \mathcal{F}_t \vee \tilde{\mathcal{F}}_T \right\}}{E\left\{ \frac{d\mathbb{Q}}{d\mathrm{P}} \middle| \mathcal{F}_t \right\}} = \exp(\tilde{R}_t^T), \ \ t \in [0, T], \tag{4.2}$$
is a continuous $(\mathcal{F}_t \vee \tilde{\mathcal{F}}_T)$-semimartingale.

Note that $\tilde{R}_t^T$ is not necessarily $\mathcal{F}_t$-measurable but that $E\left\{\exp(\tilde{R}_t^T)|\mathcal{F}_t\right\} = 1$. Moreover, $\{\tilde{\mathcal{F}}_t\}$ is idiosyncratic up to time T if an only if it has zero price. Unless explicitly stated otherwise, prices of information structures will always refer to time T and equivalent martingale measure $\mathbb{Q}$. Notice that by assuming that $\tilde{R}_t^T$ is somehow fixed, one reduces the set of equivalent martingale measures $\mathbb{Q}$. As a consequence of such a reduction pricing by arbitrage might become possible as already shown in the previous section where we imposed $\tilde{R}_t^T = 0$. Before showing this for the general case, let us first generalize Lemma 2.2 to the case of information structures with possibly non-zero price, i.e. systematic information.

Lemma 4.2 *Suppose that X is measurable with respect to $\mathcal{F}_t \vee \tilde{\mathcal{F}}_T$ where $\{\tilde{\mathcal{F}}_t\}$ has price $\tilde{R}_t^T$. Then*

$$E_{\mathbb{Q}}\{X|\mathcal{F}_t\} = E\{X \exp(\tilde{R}_t^T)|\mathcal{F}_t\}.$$

PROOF: The proof follows the same lines as the proof of Lemma 2.2. □

Before presenting our main theorem, we give one extra definition.

Definition 4.3 *An information structure $\{\tilde{\mathcal{F}}_t : \ t \geq 0\}$ is said to complete the market up to time T for X if*

$$E_{\mathbb{Q}}\left\{X|\, \mathcal{F}_t \vee \tilde{\mathcal{F}}_T\right\},$$

is constant over the set of all equivalent martingale measures.

Essentially it was shown in Section 3, under the conditions stated there, that the information in the stochastic volatility completes the market for the call option.

For notational convenience we return to the setup at the beginning of Section 3. Thus we have available an asset S, a bond B, and a money market account M. Again this does not exclude the possibility that other assets are available in the economy. We have the following simple but useful theorem.

Theorem 4.4 *Let $\tilde{\mathcal{F}}_t$ be any information structure with price $\tilde{R}_t^T$ which completes the market for $\max\{0, S_T - K\}/M_T$. Then the time t price of a European option with expiration date T and exercise price K is given by*

$$E\left\{M_t E_{\mathbb{Q}}\left\{\max\{0, S_T - K\}/M_T|\, \mathcal{F}_t \vee \tilde{\mathcal{F}}_T\right\} \exp(\tilde{R}_t^T)\Big|\, \mathcal{F}_t\right\},$$

where $\mathbb{Q}$ is any equivalent martingale measure [subject to (4.2)].

PROOF: The proof is straightforward using (4.1). □

Thus, if the reduction of the set of equivalent martingale measures by fixing $\tilde{R}_t^T$ completes the market, pricing by arbitrage becomes possible. Theorem 4.4 gives the resulting price of a European option.

It is useful to illustrate Theorem 4.4 by some examples. But before we go into these observe that our option pricing procedure is able to handle all cases where pricing by arbitrage is possible. Obviously, if pricing by arbitrage is already possible in the original economy the choice $\tilde{\mathcal{F}}_t = \{\emptyset, \Omega\}$ does the trick. This trivial information structure has zero price, i.e. is idiosyncratic and it completes the market, according to Definition 4.3, since the market was already complete. Let us now turn to other examples in which markets are incomplete.

The following example reconsiders the stochastic volatility case, but now without the idiosyncraticy assumption of Example 3.2.

Example 4.1 Assume that the instantaneous interest rate r is constant and that the asset price process is given by

$$\begin{aligned} \mathrm{d}S_t &= \varphi(S_t, \sigma_t) S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t^{(1)}, \\ \mathrm{d}\sigma_t^2 &= \mu(\sigma_t)\sigma_t^2 \mathrm{d}t + \xi(\sigma_t)\sigma_t^2 \mathrm{d}W_t^{(2)}, \end{aligned}$$

where $W^{(1)}$ and $W^{(2)}$ are independent Brownian motions. Take $\tilde{\mathcal{F}}_t = \sigma(W_s^{(2)} : 0 \le s \le t)$. Assume that the price of the stochastic volatility $\tilde{R}_t^T$, for each $t \in [0, T]$, is given by

$$\tilde{R}_t^T = \int_t^T \lambda_s \mathrm{d}W_s^{(2)} - \frac{1}{2}\int_t^T \lambda_s^2 \mathrm{d}s,$$

where λ_s defines a $\{\tilde{\mathcal{F}}_s\}$-progressively measurable process satisfying $\int_0^T \lambda_t^2 \mathrm{d}t < \infty$ $(a.s)$. Note that this implies that $\langle S, \tilde{R}^T \rangle_t = 0$. Then the price of a European call option with expiration date T and exercise price K is given by

$$E\{BS(S_t; \int_t^T \sigma_s^2 \mathrm{d}s, K \exp(-r(T-t))) \exp(\tilde{R}_t^T) | \mathcal{F}_t\}.$$

The set of equivalent martingale measures $\mathbb{Q}$ in this example is the same as in Example 3.2. However, the present choice for $\tilde{R}^T$ fixes some process λ_t, not necessarily $\lambda_t = 0$. The present example provides a considerable extension of Example 3.2. □

The method of proof used in Section 3 essentially involves enlarging the information structure so as to obtain a complete market and such that discounted asset prices remain martingales, under the equivalent martingale measure, for the enlarged filtration. In the more general setup of the present section this latter condition cannot be always met. Therefore we will need to know how continuous $\mathbb{Q}$-local martingales for $\{\mathcal{F}_t\}$ behave under $\{\mathcal{F}_t \vee \tilde{\mathcal{F}}_t\}$. The following lemma gives a partial answer which suffices for our needs.

Lemma 4.5 *Consider a filtration $\{\tilde{\mathcal{F}}_t\}$ which has time T price $\tilde{R}_t^T$ under $\mathbb{Q}$. Let X be both a $(\mathbb{P}, \mathcal{F}_t)$ and a $(\mathbb{P}, \mathcal{F}_t \vee \tilde{\mathcal{F}}_T)$ continuous semimartingale with the same decomposition for both filtrations. Then X is a continuous $(\mathbb{Q}, \mathcal{F}_t)$ local martingales if and only if $X + \langle X, \tilde{R}^T \rangle$ is a continuous $(\mathbb{Q}, \mathcal{F}_t \vee \tilde{\mathcal{F}}_T)$ local martingale.*

Proof: The proof is a straightforward application of Girsanov's Theorem together with Definition 4.1. Write [using, e.g., Revuz and Yor (1991), Proposition (VIII).1.6]

$$\begin{aligned} E\left\{ \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \middle| \mathcal{F}_t \right\} &= \mathcal{E}(L)_t, \\ E\left\{ \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \middle| \mathcal{F}_t \vee \tilde{\mathcal{F}}_T \right\} &= \mathcal{E}(\tilde{L})_t, \end{aligned}$$

and note that $\tilde{R}_t^T = \tilde{L}_t - L_t - \frac{1}{2}\langle \tilde{L}, \tilde{L} \rangle_t + \frac{1}{2}\langle L, L \rangle_t$. If X is a continuous $(\mathbb{Q}, \mathcal{F}_t)$ local martingale, then by Girsanov's Theorem, we know that $X - \langle X, L \rangle$ is the continuous

$(\mathbb{P}, \mathcal{F}_t)$ local martingale part of X and hence is a continuous $(\mathbb{P}, \mathcal{F}_t \vee \tilde{\mathcal{F}}_T)$ local martingale. Using Girsanov's Theorem once more yields that $X - \langle X, L\rangle + \langle X, \tilde{L}\rangle = X + \langle X, \tilde{R}^T\rangle$ is a continuous $(\mathbb{Q}, \mathcal{F}_t \vee \tilde{\mathcal{F}}_T)$ local martingale. The other implication is proved along the same lines. □

It will be clear from the proof of this lemma, that we could easily accommodate information structures that do affect the semimartingale decomposition of the price processes under $\mathbb{P}$. However, we will have no need for this result. Observe that this lemma was implicitly used in the proof of Theorem 3.1 with idiosyncratic $\{\tilde{\mathcal{F}}_t\}$, i.e. with $\tilde{R}_t^T = 0$. Moreover, observe that in this case of idiosyncratic information the adjustment term $\langle X, \tilde{R}^T\rangle_t$ is zero.

The next example shows how to circumvent the assumption that the stochastic volatility process may not be influenced by the asset price process. For expository reasons we return for the moment to idiosyncratic information.

Example 4.2 Consider again the price process of Hull and White (1987), where now the Brownian motions governing asset price and stochastic volatility may be correlated and the asset price may influence the stochastic volatility. That is, we consider

$$
\begin{aligned}
r_t &= r(S_t, \sigma_t), \\
\mathrm{d}S_t &= \varphi(S_t, \sigma_t) S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t^{(1)}, \\
\mathrm{d}\sigma_t^2 &= \mu(S_t, \sigma_t)\sigma_t^2 \mathrm{d}t + \xi(S_t, \sigma_t)\sigma_t^2 \mathrm{d}\left[\rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(2)}\right],
\end{aligned}
$$

where $W^{(1)}$ and $W^{(2)}$ are independent Brownian motions. Assume that the information in $\tilde{\mathcal{F}}_t = \sigma\left(W_s^{(2)} :\ 0 \le s \le t\right)$ is idiosyncratic. Given that $\{\tilde{\mathcal{F}}_t\}$ has price zero, Lemma 4.5 implies that, conditional on $\tilde{\mathcal{F}}_T$, we must have[5]

$$
\begin{aligned}
r_t &= r(S_t, \sigma_t), \\
\mathrm{d}S_t &= r(S_t, \sigma_t) S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}V_t, \\
\mathrm{d}V_t &= \mathrm{d}W_t^{(1)} + \frac{\varphi(S_t, \sigma_t) - r(S_t, \sigma_t)}{\sigma_t}\mathrm{d}t, \\
\mathrm{d}\sigma_t^2 &= \mu(S_t, \sigma_t)\sigma_t^2 \mathrm{d}t + \xi(S_t, \sigma_t)\sigma_t^2 \mathrm{d}\left[\rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(2)}\right],
\end{aligned}
$$

where V is a Brownian motion under $\mathbb{Q}$. Solving these equations and taking expectations conditional on $\mathcal{F}_t \vee \tilde{\mathcal{F}}_T$, yields a unique value for the inner expectation in (4.1). Taking the expectation under $\mathbb{P}$ conditional on $\mathcal{F}_t$ of this result gives the option price according to Theorem 4.4 ($\tilde{R}_t^T = 0$).

[5] A warning applies at this point. Conditional on $\tilde{\mathcal{F}}_T$, $W^{(2)}$ is no longer a Brownian motion and hence the stochastic differential equations are no longer defined. It is understood that the equations are solved for the filtration $\mathcal{F}_t$ and the resorting process is studied under $\mathcal{F}_t \vee \tilde{\mathcal{F}}_T$.

Notice that the set of equivalent martingale measures $\mathbb{Q}$ in this example is the same as in Examples 3.1 and 4.1. The assumption that the information in $\tilde{\mathcal{F}}_t$ is idiosyncratic forces $\lambda_t = 0$ for all $t \in [0,T]$ $(a.s)$. $\Box$

Observe that in this case it is not necessarily possible to reduce the dependence on (the distribution of) future interest rates to the price of a pure discount bond with the same maturity [compare, e.g., Amin and Ng (1993)].

The following example shows how to handle square-root like processes including non-idiosyncratic information structures. Moreover, this is an example where $\langle S, \tilde{R}^T \rangle_t$ is not necessarily zero.

Example 4.3 Consider the following stock price process

$$\begin{aligned} \mathrm{d}S_t &= \varphi(S_t, \sigma_t) S_t \mathrm{d}t + \sigma_t \sqrt{S_t} \mathrm{d}W_t^{(1)}, \\ \mathrm{d}\sigma_t^2 &= \mu(\sigma_t)\sigma_t^2 \mathrm{d}t + \xi(\sigma_t)\sigma_t^2 \mathrm{d}W_t^{(2)}, \end{aligned} \tag{4.3}$$

where $W^{(1)}$ and $W^{(2)}$ are independent Brownian motions. For simplicity of notation we assume that the interest rate is constant. We assume that the price of $\tilde{\mathcal{F}}_t = \sigma\left(W_s^{(2)} :\ 0 \le s \le t\right) = \sigma\left(\sigma_s^2 :\ 0 \le s \le t\right)$ is given by

$$\tilde{R}_t^T = \lambda(S_t)(W_T^{(2)} - W_t^{(2)}) - \frac{1}{2}\lambda^2(S_t)(T-t), \tag{4.4}$$

where $\lambda : \mathbb{R} \to \mathbb{R}$ is differentiable. Note that $\mathrm{d}\langle S, \tilde{R}^T \rangle_t = (W_T^{(2)} - W_t^{(2)} - (T - t)\lambda(S_t))\lambda'(S_t)\sigma_t^2 S_t \mathrm{d}t$. Let $CIR(S_t; \sigma, a, b)$ denote the option price at time t for a complete market in which, under the unique equivalent martingale measure,

$$\mathrm{d}S_t = [r + (a - b\lambda(S_t))\lambda'(S_t)\sigma_t^2] S_t \mathrm{d}t + \sigma_t \sqrt{S_t} \mathrm{d}W_t,$$

with σ_t a deterministic function of time, cf. Cox, Ingersoll, and Ross (1985). Note that Cox, Ingersoll, and Ross (1985) consider the term structure of interest rates, while we use their square-root model to describe stock prices in an incomplete market. Furthermore, the function CIR defined above is generally difficult to obtain analytically, in which case numerical techniques from complete market option pricing have to be used in order to evaluate CIR.

Now the option price in the described market is given by

$$E\left\{ CIR(S_t; \sigma, W_T^{(2)} - W_t^{(2)}, T-t) \exp(\tilde{R}_t^T) \Big| \mathcal{F}_t \right\},$$

where $\tilde{R}_t^T$ is given by (4.4) and σ by (4.3).

The set of equivalent martingale measures is now equal to

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}(X)_T \mathcal{E}(Y_\lambda)_T,$$

where

$$\begin{aligned} \mathrm{d}X_t &= \kappa_t \mathrm{d}W_t^{(1)}, \ \kappa_t = -\frac{\varphi(S_t,\sigma_t)S_t - rS_t}{\sigma_t\sqrt{S_t}}, \\ \mathrm{d}Y_{\lambda t} &= \lambda_t \mathrm{d}W_t^{(2)}, \end{aligned}$$

with λ_t progressively measurable satisfying $\int_0^T \lambda_t^2 \mathrm{d}t < \infty$ $(a.s)$. The choice (4.4) for $\tilde{R}_t^T$ fixes some λ_t, so that pricing by arbitrage becomes possible. □

It should be obvious from the previous results that accommodating stochastic interest rates or dependent Brownian motions is only a matter of more complicated notation and calculus, but not conceptually different. Note that the option pricing formula derived in the former example is again of the "expected complete markets formula"-type. By now it should be clear that this generally occurs as a consequence of our approach to the option pricing problem.

5 Conclusions and possible extensions

This paper discusses option pricing in incomplete markets. Since in incomplete markets pricing by arbitrage is generally not possible, additional assumptions must be made to obtain explicit option pricing formulae. We formulate these extra assumptions in terms of the price of the information structure that induces the incompleteness. By fixing this information structure price, the set of equivalent martingale measures may be reduced sufficiently well in order to make pricing by arbitrage possible. This is shown in many examples.

As a first step we considered option pricing with idiosyncratic (i.e. zero price) stochastic volatility. As in Hull and White (1987) the option price becomes an expectation involving the standard Black-Scholes formula. Our proof of this result suggests a general approach to the problem of option pricing in incomplete markets. This leads to a general option pricing method which always yields a formula of the "expected complete markets formula"-type.

One of the maintained assumptions throughout this paper is that all processes are continuous i.e. do not exhibit jumps. Generally, jumps in the price process of an asset induces incompleteness of the market, since the jumps cannot be hedged perfectly. Although we did not pursue this, we expect that the methodology described in this paper could also handle processes with jumps, the problem of course being that the mathematics involved becomes more complicated.

A Appendix

In this appendix we recall a lemma which is used in the proof of Theorem 3.1. The proof of the lemma requires a well-known result which has previously been used to solve the same problem by Amin and Ng (1993).

Lemma A.1 *If*

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} ; \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix} \right),$$

then

$$E \max\{0, e^X - e^Y\} = BS(e^{\mu_X + \frac{1}{2}\sigma_X^2}; \sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}, e^{\mu_Y + \frac{1}{2}\sigma_Y^2}),$$

where BS is defined in (3.3).

PROOF: Note that we need to calculate

$$Ee^{Y - \mu_Y - \frac{1}{2}\sigma_Y^2} \max\{0, e^{X - Y + \mu_Y + \frac{1}{2}\sigma_Y^2} - e^{\mu_Y + \frac{1}{2}\sigma_Y^2}\}.$$

If we use $\exp(Y - \mu_Y - \frac{1}{2}\sigma_Y^2)$ as a Radon-Nikodym derivative to define an expectation operator $\tilde{E}$, then we have under $\tilde{E}$ that $X - Y + \mu_Y + \frac{1}{2}\sigma_Y^2$ has the following normal distribution

$$N(\mu_X - \frac{1}{2}\sigma_Y^2 + \sigma_{XY}; \sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}).$$

Given this the result follows easily. $\square$

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